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$\mathbb{P}_{max}^{\mathfrak{d}=\aleph_1}$ and other variations

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1 Introduction of $\mathbb{P}_{max}$ variations

$\mathbb{P}_{max}$ has been introduced by W. Hugh Woodin who says that in [11], $\mathbb{P}_{max}$ forces the *canonical model* of the negation of the Continuum Hypothesis CH over $L(\mathbb{R})$ with some large cardinal assumptions, e.g. $\mathsf{AD}^{L(\mathbb{R})}$, or there are infinitely many Woodin cardinals with the measurable cardinal above. Under suitable large cardinal assumptions (in this paper, I abbreviate this to LC), $\mathbb{P}_{max}$ generically adds, over $L(\mathbb{R})$, a directed system of countable transitive models of ZFC (or its fragments) whose limit restricted to $H(\omega_2)$ (in this extension) is the whole $H(\omega_2)$, and $\mathbb{P}_{max}$ forces that the nonstationary ideal NS_{ω_1} on ω_1 is saturated. One of the important facts on $\mathbb{P}_{max}$ is absoluteness of Π_2-sentences for the structure

$$\langle H(\omega_2), \in, NS_{\omega_1}, R\rangle$$

for some set R of reals in $L(\mathbb{R})$ as follows:

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> If a Π_2-sentence for the structure $\langle H(\omega_2), \in, NS_{\omega_1}, R\rangle$ is Ω_{ZFC}-consistent (e.g. forceable by set-forcing over ZFC), then it is true in $\langle H(\omega_2), \in, NS_{\omega_1}, R\rangle$ in the extension with $\mathbb{P}_{max}$ over $L(\mathbb{R})$ with LC.

(Under LC (e.g. there exist proper class many Woodin cardinals), every set of reals in $L(\mathbb{R})$ is universally Baire, and weakly homogeneously Suslin (see e.g. [5]). R is considered as an interpretation of its universally Baire set of reals in each universe. For more historical and technical remarks on $\mathbb{P}_{max}$, see [11, 7, 1].)

In [11], Woodin studied not only $\mathbb{P}_{max}$ but also conditional variations of $\mathbb{P}_{max}$ for e.g. Suslin trees and the Borel Conjecture. $\mathbb{P}_{max}$ variations have been studied by several set theorists: Feng–Woodin, Larson, Larson–Todorčević, Shelah–Zapletal and Yorioka [3, 4, 6, 8, 10, 12]. In [10], many variations of $\mathbb{P}_{max}$ for Σ_2-statements in the structure $H(\omega_2)$ on cardinal invariants of the reals have been investigated. We should notice that all of them are derived from $\diamondsuit$. For example, the $\mathbb{P}_{max}$ variation, say $\mathbb{P}_{max}^{\mathfrak{d}=\aleph_1}$, for the statement that the dominating number $\mathfrak{d}$ in ω^ω is $\aleph_1$ has been studied. It has been proved in [10, §2] that the extension with $\mathbb{P}_{max}^{\mathfrak{d}=\aleph_1}$ over $L(\mathbb{R})$ under LC satisfies ZFC, the continuum $\mathfrak{c}$ is $\aleph_2$, NS_{ω_1} is saturated, $\mathfrak{d} = \aleph_1$ holds, and maximality with respect to Π_2-statements in $\langle H(\omega_2), \in, NS_{\omega_1}, R\rangle$ for some set R of reals in $L(\mathbb{R})$, that is, under LC, the extension with $\mathbb{P}_{max}^{\mathfrak{d}=\aleph_1}$ over $L(\mathbb{R})$ satisfies the following property, called Π_2-*compactness* in [10]:

> If ψ is a Π_2-sentence for the structure $\langle H(\omega_2), \in, NS_{\omega_1}, R\rangle$ and the statement $\langle H(\omega_2), \in, NS_{\omega_1}, R\rangle \models$"$\mathfrak{d} = \aleph_1 \wedge \psi$" is Ω_{ZFC}-consistent, then it is true in $\langle H(\omega_2), \in, NS_{\omega_1}, R\rangle$.

So this model can be considered as the canonical model of $\mathfrak{d} = \aleph_1$. In [10], there are many examples and counterexamples of Π_2-compact statements. One non-Π_2-compact statement, which does not appear in [10], is that the additivity $\mathsf{add}(\mathcal{M})$ of the meager ideal is $\aleph_1$: By Miller–Truss's characterization of $\mathsf{add}(\mathcal{M})$, $\mathsf{add}(\mathcal{M})$ is the minimum of the bounding number $\mathfrak{b}$ and the covering number $\mathsf{cov}(\mathcal{M})$ of the meager ideal. However both "$\aleph_1 = \mathsf{add}(\mathcal{M}) < \mathfrak{b}$" and "$\aleph_1 = \mathsf{add}(\mathcal{M}) < \mathsf{cov}(\mathcal{M})$" are consistent with ZFC, and both "$\mathsf{cov}(\mathcal{M}) > \aleph_1$" and "$\mathfrak{b} > \aleph_1$" are Π_2-statements in the structure $\langle H(\omega_2), \in, NS_{\omega_1}\rangle$. (The statement that the additivity of the null ideal is $\aleph_1$ is not Π_2-compact either. It is known that $\mathsf{add}(\mathcal{N}) = \min\{\mathsf{add}^*(\mathcal{N}), \mathfrak{b}\}$. See [2, Theorem 2.7.13.] or [9].)

In this paper, we work in ZFC except for the definition of $\mathbb{P}^{\phi}_{max}$ and the proof of Theorem Schemes because when we force by $\mathbb{P}^{\phi}_{max}$, we always consider $L(\mathbb{R})$ as the ground model which never satisfies the Axiom of Choice (by our assumption). $\mathbb{P}_{max}$ can be defined by various ways. One of them is defined by use of *iterable pairs*. Suppose a suitable large cardinal property, M is a countable transitive model of ZFC and I is a member of M which is a uniform normal ideal on ${\omega_1}^M$ in M. We can take a direct system $\langle M_\gamma, G_\beta, j_{\gamma,\delta}; \beta < \gamma \leq \delta \leq \omega_1 \rangle$, called an *iteration* of (M, I) (of length ω_1), such that

- $M_0 = M$,
- G_β is an M_β-generic filter of the forcing notion $\left(\mathcal{P}\left({\omega_1}^{M_\beta}\right)/j_{0,\beta}(I)\right)^{M_\beta}$ (or $\left(\mathcal{P}\left({\omega_1}^{M_\beta}\right)\setminus j_{0,\beta}(I)\right)^{M_\beta}$) for every $\beta \in \omega_1$,
- $j_{\gamma,\gamma}$ is the identity on M_γ for every $\gamma \in \omega_1 + 1$,
- $M_{\beta+1}$ is (the transitive collapse of) the generic ultrapower of M_β by G_β (if it is wellfounded, otherwise we stop the construction), and $j_{\gamma,\gamma+1}$ is the ultrapower embedding induced by G_γ for every $\gamma \in \omega_1$, and
- if $\alpha \in \omega_1 + 1$ is a limit ordinal, then M_α is (the transitive collapse of) the direct limit of the system $\langle M_\gamma, j_{\gamma,\delta}; \gamma \leq \delta < \alpha \rangle$ and $j_{\gamma,\alpha}$ is the induced embedding for every $\gamma \in \alpha$.

(See [11, Definition 3.5. or Definition 4.1.] or [7, 1.2 Definition].) A pair (M, I) as above is called iterable if all M_γ, $\gamma \in \omega_1$, are wellfounded regardless of the choice of generic filters G_β. Woodin proved that if I is precipitous, then (M, I) is iterable (see [11, Lemma 3.10. and Lemma 4.5.]).

In many cases, we define the $\mathbb{P}_{max}$ variation $\mathbb{P}^{\phi}_{max}$ for a Σ_2-sentence ϕ in the structure $\langle H(\omega_2), \in, NS_{\omega_1} \rangle$ which is derived from $\Diamond$. For example, $\mathfrak{d} = \aleph_1$ holds, and there exists a coherent Suslin tree, etc. In [10], variations of $\mathbb{P}_{max}$ are defined by use of stationary tower forcing ([5]). In this paper, we adopt a definition in [7, §10.2], however all of proofs in this paper can be applied to any type of $\mathbb{P}^{\phi}_{max}$ variations.

Definition of $\mathbb{P}^{\phi}_{max}$ ([7, §10.2]) *Let ϕ be a Σ_2-statement for the structure $\langle H(\omega_2), \in, NS_{\omega_1} \rangle$, and say that ϕ forms $\exists u \forall v\, \phi_0(u, v)$. Conditions of the forcing notion $\mathbb{P}^{\phi}_{max}$ are defined by recursion on their ranks as follows. A triple $\langle (M, I), a, \mathcal{X} \rangle$ is a condition of $\mathbb{P}_{max}$ if*

1. (M, I) *is an iterable pair,*
2. $a \in H(\omega_2)^M$ *and* $\langle H(\omega_2), \in, I\rangle^M \models$"$\forall v\, \phi_0(a, v)$ ", *and*
3. $\mathcal{X}$ *is a member of M and a set (possibly empty) of pairs* $\langle\langle(N, J), b, \mathcal{Y}\rangle, j\rangle$ *such that*
   - $\langle(N, J), b, \mathcal{Y}\rangle \in \mathbb{P}^{\phi}_{max} \cap H(\omega_1)^M$,
   - j *is in* M *and an iteration of* (N, J) *of length* ${\omega_1}^M$ *such that* $j(J) = I \cap j\left(\mathcal{P}\left({\omega_1}^N\right)^N\right)$, $j(b) = a$ *and* $j(\mathcal{Y}) \subseteq \mathcal{X}$, *and*
   - $\mathcal{X}$ *forms a function, i.e. for members* $\langle p, j\rangle$ *and* $\langle p', j'\rangle$ *in* $\mathcal{X}$, *if* $p = p'$, *then* $j = j'$.

For conditions $\langle(M, I), a, \mathcal{X}\rangle$ *and* $\langle(N, J), b, \mathcal{Y}\rangle$ *in* $\mathbb{P}^{\phi}_{max}$, *we define*

$$\langle(M, I), a, \mathcal{X}\rangle <_{\mathbb{P}^{\phi}_{max}} \langle(N, J), b, \mathcal{Y}\rangle$$

if there exists j *such that* $\langle\langle(N, J), b, \mathcal{Y}\rangle, j\rangle \in \mathcal{X}$.

We have to note that the statement that a pair (M, I) is iterable is Π^1_2 about a real coding (M, I), so is absolute (see e.g. [7, 1.3 Remark and 1.10 Remark]). Therefore the statement that a triple $\langle(M, I), a, \mathcal{X}\rangle$ is a condition of $\mathbb{P}^{\phi}_{max}$ is also Π^1_2, and so is absolute. Since $L(\mathbb{R})$ has every real, it also has every countable transitive model. And since a condition of $\mathbb{P}^{\phi}_{max}$ can be coded by a real, $\left(\mathbb{P}^{\phi}_{max}\right)^{L(\mathbb{R})} = \mathbb{P}^{\phi}_{max}$. If ϕ is trivial (e.g. " $0 = 0$ ", or the statement that there exists the empty set), then $\mathbb{P}^{\phi}_{max}$ can be considered the standard $\mathbb{P}_{max}$. (However $\mathbb{P}^{\phi}_{max}$ and $\mathbb{P}_{max}$ are slightly different, see [11, §5.4, in particular Theorem 5.40.].)

To analyze the extension by $\mathbb{P}^{\phi}_{max}$, we need some game theoretic lemmata. (On definitions of games $\mathcal{G}^{\phi}_1$, $\mathcal{G}^{\phi}_{\omega}$ and $\mathcal{G}^{\phi}_{\omega_1}$, I refer [7, §3 and §10.2].)

We define the game $\mathcal{G}^{\phi}_1$ as follows. Suppose that $\langle(M, I), a, \mathcal{X}\rangle$ is a condition of $\mathbb{P}^{\phi}_{max}$, J is a normal uniform ideal on ω_1. Players **I** and **II** collaborate to build an iteration $\langle M_\gamma, G_\beta, j_{\gamma,\delta}; \beta < \gamma \leq \delta \leq \omega_1\rangle$ of (M, I) with the following rule: In each round α, **II** chooses a set A in the set $\mathcal{P}\left({\omega_1}^{M_\alpha}\right)^{M_\alpha} \setminus j_{0,\alpha}(I)$, and then **I** chooses an $\left(M_\alpha, \left(\mathcal{P}\left({\omega_1}^{M_\alpha}\right) \setminus j_{0,\alpha}(I)\right)^{M_\alpha}\right)$-generic filter G_α with $A \in G_\alpha$. (To just simplify notation, we force by $\mathcal{P}(\omega_1) \setminus I$ instead of $\mathcal{P}(\omega_1)/I$ in this paper.) After all ω_1 many rounds have been played, **I** wins if

- $\langle H(\omega_2), \in, J\rangle \models$"$\forall v\, \phi_0(j_{0,\omega_1}(a), v)$ ".

(We should note that player **II** has a strategy such that after all ω_1 rounds have been played whenever player **II** plays according to this strategy,

- $j_{0,\omega_1}(I) = J \cap M_{\omega_1}$ holds.

See [11, Lemma 4.36.], [7, 2.8 Lemma], [1, Lemma 1.8].)

To show σ-closedness of $\mathbb{P}^{\phi}_{max}$ and define the strategic iteration lemma for ϕ, we need to define an iterable limit sequence and two games $\mathcal{G}^{\phi}_{\omega}$ and $\mathcal{G}^{\phi}_{\omega_1}$. (On this paragraph, see [11, Chapter 4.1 and Lemma 4.43.], [7, §3] and [1, §2].) Let $\langle p_i; i \in \omega\rangle$ is a decreasing sequence of $\mathbb{P}^{\phi}_{max}$ and write $p_i := \langle (M_i, I_i), a_i, \mathcal{X}_i\rangle$. Let $j_{i,i+1} : (M_i, I_i) \to (M_i^*, I_i^*)$ be an iteration witnessing that $p_{i+1} <_{\mathbb{P}^{\phi}_{max}} p_i$ (and if $p_{i+1} = p_i$, then let $j_{i,i+1}$ be the identity map) and let $\{j_{i,i'}; i \leq i' \leq \omega\}$ be the commuting family of embeddings generated by $\{j_{i,i+1}; i \in \omega\}$. We write $j_{i,\omega}; (M_i, I_i) \to (N_i, J_i)$ for each $i \in \omega$. Let $a := \bigcup_{i\in\omega} a_i$ and $\mathcal{X} := \bigcup_{i\in\omega} \mathcal{X}_i$. In most cases, a forms a witness of ϕ in every N_i. (At least, every application in any present published paper, including this paper, on $\mathbb{P}^{\phi}_{max}$ and its variations is in this case.) Then we can show that

- for each $i \in \omega$, (N_i, J_i) is an iterable pair,
- for each $i \in \omega$, $N_i \in N_{i+1}$ and ${\omega_1}^{N_i} = {\omega_1}^{N_0}$,
- for each $i \in \omega$, $J_{i+1} \cap N_i = J_i$,
- $a \in H(\omega_2)^{N_0}$ and for each $i \in \omega$, $\langle H(\omega_2), \in, J_i\rangle^{N_i} \models$"$\forall b\, \phi_0(a, b)$ ".

We call $\langle \langle (N_i, J_i); i \in \omega\rangle, a, \mathcal{X}\rangle$ a limit sequence if it is constructed as above. For a limit sequence $\langle \langle (N_i, J_i); i \in \omega\rangle, a, \mathcal{X}\rangle$, when an ultrafilter G on the set

$$\bigcup_{i\in\omega} \mathcal{P}\left({\omega_1}^{N_i}\right)^{N_i} \setminus J_i$$

satisfies that for every regressive function f on ${\omega_1}^{N_i}$ in $\bigcup_{i\in\omega} N_i$, f is constant on some condition in G, we call it a $\bigcup\{N_i; i \in \omega\}$-normal ultrafilter for $\langle \langle (N_i, J_i); i \in \omega\rangle, a, \mathcal{X}\rangle$. Then we form the ultrapower of $\langle \langle (N_i, J_i); i \in \omega\rangle, a, \mathcal{X}\rangle$ formed from G and all functions $f : {\omega_1}^{N_0} \to N_i$ in $\bigcup_{i\in\omega} N_i$. (More precisely, see [11, Definition 4.15.].) Using this ultrapower, we define the iteration of the sequence $\langle \langle (N_i, J_i); i \in \omega\rangle, a, X\rangle$, and the iterability of the sequence $\langle \langle (N_i, J_i); i \in \omega\rangle, a, \mathcal{X}\rangle$ as in the iterable pair. We note that for a limit sequence $\langle \langle (N_i, J_i); i \in \omega\rangle, a, \mathcal{X}\rangle$ constructed as above,

- $\langle\langle(N_i, J_i); i \in \omega\rangle, a, \mathcal{X}\rangle$ is iterable.

We define the game $\mathcal{G}^{\phi}_{\omega}$ as follows. Suppose that $\langle\langle(N_i, J_i); i \in \omega\rangle, a, \mathcal{X}\rangle$ is a limit sequence, J is a normal uniform ideal on ω_1. Players **I** and **II** collaborate to build an iteration of $\langle\langle(N_i, J_i); i \in \omega\rangle, a, \mathcal{X}\rangle$ consisting of limit sequences $\langle\langle(N_i^{\alpha}, J_i^{\alpha}); i \in \omega\rangle, a^{\alpha}, \mathcal{X}^{\alpha}\rangle$, $\bigcup\{N_i^{\alpha}; i \in \omega\}$-normal ultrafilters G_{α} for $\langle\langle(N_i^{\alpha}, J_i^{\alpha}); i \in \omega\rangle, a^{\alpha}, \mathcal{X}^{\alpha}\rangle$ and a commuting family of embeddings $j_{\alpha,\beta}$ for $\alpha \leq \beta \leq \omega_1$ with the following rule: In each round α, **II** chooses a set A in the set $\bigcup\left\{\mathcal{P}\left({\omega_1}^{N_i^{\alpha}}\right)^{N_i^{\alpha}} \setminus J_i^{\alpha}; i \in \omega\right\}$, and then **I** chooses a $\bigcup\{N_i^{\alpha}; i \in \omega\}$-normal ultrafilter G_{α} for $\langle\langle(N_i^{\alpha}, J_i^{\alpha}); i \in \omega\rangle, a^{\alpha}, \mathcal{X}^{\alpha}\rangle$ with $A \in G_{\alpha}$. After all ω_1 many rounds have been played, **I** wins if

- $\langle H(\omega_2), \in, J\rangle \models$"$\forall v\, \phi_0(j_{0,\omega_1}(a), v)$ ".

(We should note that **II** has a strategy such that after all ω_1 rounds have been played whenever player **II** plays according to this strategy,

- $J_i^{\omega_1} = J \cap N_i^{\omega_1}$ holds for every $i \in \omega$.

We can prove σ-closedness of $\mathbb{P}^{\phi}_{max}$ using strategies for both players **I** and **II**. See [11, Lemma 4.43.], [7, 3.4 Lemma and 3.5 Lemma], [1, Lemma 2.5].)

We define the game $\mathcal{G}^{\phi}_{\omega_1}$ as follows. Let p_0 is a condition of $\mathbb{P}^{\phi}_{max}$. Players **I** and **II** collaborate to build a decreasing ω_1-chain $\langle p_{\alpha}; \alpha \in \omega_1\rangle$ of conditions with the following rule: In each round α, if α is a successor ordinal, **II** chooses a condition p_{α} below $p_{\alpha-1}$. If α is a limit ordinal, then **II** chooses a cofinal ω-sequence of α and, letting $\langle\langle(N_i^{\alpha}, J_i^{\alpha}); i \in \omega\rangle, a_{\alpha}^{*}, \mathcal{X}_{\alpha}^{*}\rangle$ be the induced limit sequence, **II** chooses a set A_{α} in the set $\bigcup\left\{\mathcal{P}\left({\omega_1}^{N_i^{\alpha}}\right)^{N_i^{\alpha}} \setminus J_i^{\alpha}; i \in \omega\right\}$, and then **I** chooses a condition $p_{\alpha} = \langle(M_{\alpha}, I_{\alpha}), a_{\alpha}, \mathcal{X}_{\alpha}\rangle$ below every p_{β} such that for some iteration k of $\langle\langle(N_i^{\alpha}, J_i^{\alpha}); i \in \omega\rangle, a_{\alpha}^{*}, \mathcal{X}_{\alpha}^{*}\rangle$, $k[\mathcal{X}_{\alpha}^{*}] \subseteq \mathcal{X}_{\alpha}$ and ${\omega_1}^{N_0^{\alpha}} \in k(A_{\alpha})$. After all ω_1 rounds have been played, **I** wins if, letting $j_{\alpha,\beta}$ $(\alpha < \beta \leq \omega_1)$ be the induced commuting family of embeddings on the sequence $\langle p_{\alpha}; \alpha \in \omega_1\rangle$,

- $\langle H(\omega_2), \in, j_{0,\omega_1}(I_0)\rangle \models$"$\forall v\, \phi_0(j_{0,\omega_1}(a), v)$ ".

(In [10], the strategic iteration lemma for ϕ is the following lemma scheme:

(ZFC $+\Diamond$) Player **I** has a winning strategy in $\mathcal{G}^{\phi}_{\omega_1}$.

This is related to [7, 5.2 Theorem].)

The following theorem is a basic theorem of $\mathbb{P}^{\phi}_{max}$.

Theorem Scheme 1 ([11, Chapter 4], [1, §§3-5], [7, §§5-7], [10, §1]) (ZFC + LC) *Let ϕ be a Σ_2-sentence in the structure $\langle H(\omega_2), \in, NS_{\omega_1}\rangle$. Assume that the following three statements*

(1) *player **I** has a winning strategy in $\mathcal{G}_1^{\phi}$,*

(ω) *player **I** has a winning strategy in $\mathcal{G}_{\omega}^{\phi}$,*

(ω_1) *player **I** has a winning strategy in $\mathcal{G}_{\omega_1}^{\phi}$,*

are all Ω_{ZFC}-consistent. Let G be a $(L(\mathbb{R}), \mathbb{P}_{max}^{\phi})$-generic filter. Then in $L(\mathbb{R})[G]$, ZFC holds, $\mathfrak{c} = \aleph_2$, NS_{ω_1} is saturated and $\langle H(\omega_2), \in, NS_{\omega_1}\rangle \models$"$\phi$" holds.

In the above theorem scheme, the phrase that (1), (ω) and (ω_1) are all Ω_{ZFC}-consistent are usually considered as the slightly stronger following statement:

(ZFC $+\Diamond$) Both (1), (ω) and (ω_1) hold.

One of important conclusions of $\mathbb{P}_{max}^{\phi}$ extensions is Π_2-maximality. To show this, we need a more technical lemma. For a sentence Φ in the language of set theory, the iteration lemma for ϕ from Φ is defined as follows:

Lemma Scheme; The Iteration Lemma for ϕ from Φ (ZFC $+\Phi$) *If*

- *(M, I) is an iterable pair,*
- *$a \in H(\omega_2)^M$ and $\langle H(\omega_2), \in, I\rangle^M \models$"$\forall b\, \phi_0(a, b)$"*
- *J is a normal uniform ideal on ω_1, and*
- *$\langle H(\omega_2), \in, J\rangle \models$"$\phi$",*

then there exists an iteration $j : (M, I) \to (M^*, I^*)$ of length ω_1 such that

- *$I^* = J \cap M^*$, and*
- *$\langle H(\omega_2), \in, J\rangle \models$"$\forall v\, \phi_0(j(a), v)$".*

Of course, the case that Φ contradicts ϕ does not make sense. In [10], the simple iteration lemma for ϕ is the iteration lemma for ϕ from $\diamondsuit$, and the optimal iteration lemma for ϕ is the iteration lemma for ϕ from any trivial statement. We note that if (under ZFC) player **I** has a winning strategy in $\mathcal{G}_1^{\phi}$, then the optimal iteration lemma for ϕ holds. We should notice that for some Σ_2-sentence ϕ in the structure $\langle H(\omega_2), \in, NS_{\omega_1}\rangle$, the simple iteration lemma for ϕ fails. For example, the simple iteration lemma for CH, and for the statement that the almost disjointness number is $\aleph_1$ fail (see [10, §1.3] and [11, Lemma 5.29.]).

Theorem Scheme 2 ([11, Chapter 4], [1, §§3-5], [7, §§5-7], [10, §1]) (ZFC + LC) *Let ϕ be a Σ_2-sentence in the structure $\langle H(\omega_2), \in, NS_{\omega_1}\rangle$ and Φ a sentence in the language of set theory such that the iteration lemma for ϕ from Φ holds. Assume that both (ω) and (ω_1) are Ω_{ZFC}-consistent. Let G be a $(L(\mathbb{R}), \mathbb{P}^{\phi}_{max})$-generic filter. Then in $L(\mathbb{R})[G]$, ZFC holds, $\mathfrak{c} = \aleph_2$, NS_{ω_1} is saturated and $\langle H(\omega_2), \in, NS_{\omega_1}\rangle \models$ "ϕ " holds, and for any Π_2-sentence ψ in the structure $\langle H(\omega_2), \in, NS_{\omega_1}, R\rangle$ for some set R of reals in $L(\mathbb{R})$, if the statement $\Phi + \langle H(\omega_2), \in, NS_{\omega_1}, R\rangle \models$ "$\phi \wedge \psi$ " is Ω_{ZFC}-consistent, then $\langle H(\omega_2), \in, NS_{\omega_1}, R\rangle \models$ "ψ " holds.*

Therefore under the assumption in Theorem Scheme 1, if the optimal iteration lemma for ϕ holds, then ϕ is Π_2-compact in the extension by $\mathbb{P}^{\phi}_{max}$ over $L(\mathbb{R})$. We have some examples of Σ_2-statements for which the optimal iteration lemma fails, e.g. for the existence of a Suslin tree. (See [10, §1.3].) However we should notice that even if the opitimal iteration lemma for ϕ fail, we *cannot* conclude that ϕ cannot be Π_2-compact.

In this note, we prove the optimal iteration lemma for $\mathfrak{d} = \aleph_1$. This proof is prototypical for any other $\mathbb{P}_{max}$ variations of $\mathfrak{x} = \aleph_1$ where $\mathfrak{x}$ is a cardinal invariant which is the smallest size of the cofinality of some ordered structure, or some ideal on the reals. The point whether we can adopt the proof for $\mathfrak{d} = \aleph_1$ to the optimal iteration lemma for $\mathfrak{x} = \aleph_1$ is whether we have a Suslin ccc Amoeba forcing for this structure and we can show the subgenericity lemma (i.e. a variation of Proposition 2.3).

2 The optimal iteration lemma for $\mathfrak{d} = \aleph_1$

We don't prove the optimal iteration lemma for $\mathfrak{d} = \aleph_1$ usually. We find an equivalent statement of $\mathfrak{d} = \aleph_1$ and we show the optimal iteration lemma for it.

Definition 2.1 ([10, Lemma 2.6.]). *Let I be a normal uniform ideal on ω_1. A sequence $\langle f_\xi; \xi \in \omega_1 \rangle$ of functions in ω^ω is an I-good scale if*

- *it is a scale, i.e. a well-ordered with respect to the eventually dominance, and*
- *for every $f \in \omega^\omega$, the set $\{\xi \in \omega_1; f_\xi$ dominates f everywhere $(f \leq f_\xi)\}$ is I-positive.*

Proposition 2.2. *Assume that I is a normal uniform ideal on ω_1. $\mathfrak{d} = \aleph_1$ holds iff there exists an I-good scale.*

Proof. Suppose that $\mathfrak{d} = \aleph_1$ holds, and let $\langle g_\xi; \xi \in \omega_1 \rangle$ be a scale, i.e.

- if $\xi < \eta$ in ω_1, then $g_\xi \leq^* g_\eta$, and
- for any $h \in \omega^\omega$, there exists $\xi \in \omega_1$ such that $h \leq^* g_\xi$.

Let

$$\langle X_{s,\alpha}; s \in \omega^{<\omega} \;\&\; \alpha \in \omega_1 \rangle$$

be a sequence of pairwise disjoint I-positive subsets of ω_1.

By recursion on $\xi \in \omega_1$, we construct $f_\xi \in \omega^\omega$ such that

- f_ξ $\leq^*$-dominates g_η and f_η for all $\eta < \xi$, and
- if ξ is in some $X_{s,\alpha}$, then f_ξ $\sqsubseteq$-dominates the function $s^\frown(g_\alpha \restriction [|s|, \infty))$.

Then we note that $\langle f_\xi; \xi \in \omega_1 \rangle$ is a scale. So what we need to check is I-goodness.

Let $f \in \omega^\omega$. Then since $\langle g_\xi; \xi \in \omega_1 \rangle$ is a scale, we can find $\alpha \in \omega_1$ so that f is $\leq^*$-dominated by g_α. Let $n \in \omega$ be such that $f(i) \leq g_\alpha(i)$ for every $i \geq n$ and let $s := f \restriction n$. Then

$$\{\xi \in \omega_1; f \leq g_\xi\} \supseteq X_{s,\alpha},$$

that is, the set $\{\xi \in \omega_1; f \leq g_\xi\}$ is I-positive.

The other direction is trivial. $\square$

We have a Suslin ccc Amoeba forcing for the structure $\langle \omega^\omega, \leq^* \rangle$, the Hechler forcing $\mathbb{D} := \omega^{<\omega} \times \omega^\omega$. For $p = \langle s^p, f^p \rangle$ and $q = \langle s^q, f^q \rangle$, $p \leq_{\mathbb{D}} q$ if $s^p \supseteq s^q$, $f^q \leq f^p$ and for every $i \in [|s^q|, |s^p|)$, $s^p(i) \geq f^q(i)$. For a condition $p \in \mathbb{D}$, we define

$$\mathsf{body}(p) := s^p{}^\frown f^p \restriction [|s^p|, \infty),$$

and let $\mathbb{D} \restriction f := \{p \in \mathbb{D}; \mathsf{body}(p) \leq f\}$.

Proposition 2.3. *Suppose that M is a model of a large enough fragment of* ZFC. *($\mathsf{ZFC} - \mathsf{Powerset} + \exists \mathcal{P}(2^\omega)$ is sufficient.) Suppose that f eventually dominates all functions in $\omega^\omega \cap M$, and $D \in M$ is such that D is dense in $\mathbb{D}$ in M. Then $D \cap (\mathbb{D} \restriction f)$ is dense in $(\mathbb{D} \restriction f) \cap M$.*

Proof. Let $p_0 = \langle s_0, f_0 \rangle \in (\mathbb{D} \restriction f) \cap M$. Working in M, we choose $p_i = \langle s_i, f_i \rangle \in \mathbb{D}$ $(\cap M)$ by induction on $i \in \omega$ such that

- $p_{i+1} \in D$, and
- $p_{i+1} \leq_{\mathbb{D}} \langle \mathsf{body}(p_0) \restriction |s_i|, f_i \rangle$. (We must note that $\langle \mathsf{body}(p_0) \restriction |s_i|, f_i \rangle$ is a condition in $\mathbb{D}$ $(\cap M)$ which extends p_0.)

Then we define $g \in \omega^\omega$ such that

$$g(i) := \begin{cases} s_0(i) & \text{if } i < |s_0| \\ s_{k+1}(i) & \text{if } |s_k| \leq i < |s_{k+1}| \text{ for some } k \in \omega \end{cases}.$$

Since $\langle p_i; i \in \omega \rangle$ is in M, g is also in M. Thus $g \leq^* f$ holds, hence for large enough $k \in \omega$, $g \restriction [|s_k|, \infty) \leq f \restriction [|s_k|, \infty)$. Then for a fixed such a k, p_{k+1} is in $D \cap (\mathbb{D} \restriction f)$. □

Corollary 2.4. *Suppose that $\mathbb{P}$ is a forcing notion and $\dot{g}$ is a $\mathbb{P}$-name such that*

1. $\Vdash_{\mathbb{P}}$ "$\dot{g} \in \omega^\omega$ & $\dot{g}$ *eventually dominates all functions in* $\omega^\omega \cap \mathbf{V}$ ", *(where* $\mathbf{V}$ *is the ground model) and*
2. *for every condition* $r \in \mathbb{SLOC}$, $\|\mathsf{body}(\check{r}) \leq \dot{g}\|_{ro(\mathbb{P})}$ *is non-zero.*

Then $\mathbb{D}$ is completely embeddable into $\mathbb{Q} := ro(\mathbb{P}) * ((\mathbb{D} \restriction \dot{g}) \cap \mathbf{V})$ such that $\Vdash_{\mathbb{Q}}$ "$\dot{f}_{G_{\mathbb{D}}} := \bigcup_{p \in G} s^p \leq \dot{g}$ ".

Proof. We show that the embedding i from $\mathbb{D}$ into $\mathbb{Q}$, defined by

$$i(r) := \left\langle \|\mathsf{body}(\check{r}) \le \dot{g}\|_{ro(\mathbb{P})}, \check{r} \right\rangle$$

for each $r \in \mathbb{D}$, is a complete embedding.

To prove this, we show that for any dense subset D in $\mathbb{D}$ (in the ground model), the set $\{i(r); r \in D\}$ is predense in $\mathbb{Q}$. Let $\langle p, \check{q} \rangle \in \mathbb{Q}$, i.e.,

$$p \Vdash_{\mathbb{P}} \text{“} \check{q} \in (\mathbb{D} \restriction \dot{g}) \cap \mathbf{V} \text{”}, \text{ i.e. } p \le_{\mathbb{P}} \|\mathsf{body}(\check{q}) \le \dot{g}\|_{ro(\mathbb{P})}.$$

Since, by the previous proposition,

$$\Vdash_{\mathbb{P}} \text{“} \check{D} \cap (\mathbb{D} \restriction \dot{g}) \text{ is dense in } (\mathbb{D} \restriction \dot{g}) \cap \mathbf{V} \text{”},$$

we can find $p' \le_{\mathbb{P}} p$ and $q' \le_{\mathbb{D}} q$ such that $q' \in D$ and

$$p' \Vdash_{\mathbb{P}} \text{“} \check{q}' \in \check{D} \cap (\mathbb{D} \restriction \dot{g}) \text{”}.$$

Then

$$\langle p', \check{q}' \rangle \le_{\mathbb{Q}} \left\langle \|\mathsf{body}(\check{q}') \le \dot{g}\|_{ro(\mathbb{P})}, \check{q}' \right\rangle = i(q').$$

Assume that i is not a complete embedding, i.e. there exists $\langle p, \check{q} \rangle$ in $\mathbb{Q}$ such that the set

$$D := \{r \in \mathbb{D}; i(r) \text{ and } \langle p, \check{q} \rangle \text{ are incompatible in } \mathbb{Q}\}$$

is dense in $\mathbb{D}$. Then the set $\{i(r); r \in D\}$ is predense in $\mathbb{Q}$. However then, there exists $r \in D$ so that $i(r)$ and $\langle p, \check{q} \rangle$ are incompatible in $\mathbb{Q}$, which is a contradiction. □

Lemma 2.5. *Suppose that M is a countable model of (a large enough fragment of)* ZFC*, $\mathbb{P}$ and $\dot{g}$ satisfy the hypothesis of Corollary 2.4 in M, $p \in \mathbb{P} \cap M$ and $f \in \omega^\omega$. (We may not assume $f \in M$.) Then there exists a $(M, \mathbb{P})$-generic filter G containing p such that f is $\le^*$-dominated by $\dot{g}[G]$.*

Proof. We fix a complete embedding from $ro(\mathbb{D})$ into $\mathbb{Q} := ro(\mathbb{P}) * ((\mathbb{D} \restriction \dot{g}) \cap \mathbf{V})$ as in the previous corollary, and let p' be a projection of p via this embedding.

Let N be a countable model of a large enough fragment of ZFC containing $M \cup \{S\}$. Since N is a countable model, there exists a $(N, \mathbb{D})$-generic filter F' containing p'. We let $F := F' \cap M$. Since $\mathbb{D}$ is a Suslin ccc forcing notion, all maximal antichains on $\mathbb{D}$ belonging to M are still maximal in N. Thus F

is $(M, \mathbb{D})$-generic and $f \leq^* f_F$. We take a $(M, \mathbb{Q})$-generic filter H extending F (via the fixed embedding) with $p \in H$ and let $G := ro(\mathbb{P}) \cap H$. We note that G is $(M, ro(\mathbb{P}))$-generic. Then

$$f \leq^* f_F \sqsubseteq \dot{g}[H] = \dot{g}[G].$$

$\square$

Theorem 2.6 (The optimal iteration lemma for the existence of a good scale). (ZFC) *If*

- *(M, I) is an iterable pair,*
- *$a \in H(\omega_2)^M$ and $H(\omega_2)^M \models$"a is an I-good scale ",*
- *J is a normal uniform ideal on ω_1, and*
- $\mathsf{cof}(\mathcal{N}) = \aleph_1$,

then there exists an iteration $j : (M, I) \to (M^*, I^*)$ of length ω_1 such that

- *$I^* = J \cap M^*$, and*
- *$j(a)$ is a J-good scale.*

Proof. Suppose that (M, I) is an iterable pair, i.e.

- M is a countable transitive model of ZFC, and
- $I \in M$ and $M \models$"I is a normal uniform ideal on ${\omega_1}^M$ ".

Let $\langle f_\xi; \xi \in {\omega_1}^M \rangle$ be in M such that

$$M \models\text{“} \langle f_\xi; \xi \in {\omega_1}^M \rangle \text{ is an } I\text{-good scale ”},$$

and $\langle g_\xi; \xi \in \omega_1 \rangle$ be a (J-good) scale. (We don't need J-goodness of the sequence $\langle g_\xi; \xi \in \omega_1 \rangle$.) Let $\langle X_{n,\alpha}; n \in \omega \;\&\; \alpha \in \omega_1 \rangle$ be a sequence of J-positive subsets of ω_1 which are pairwise disjoint.

We build an iteration $\langle M_\gamma, G_\beta, j_{\gamma,\delta}; \beta < \gamma \leq \delta \leq \omega_1 \rangle$ of (M, I) of length ω_1 such that

- for each $\alpha \in \omega_1$, we fix a sequence $\langle Y_{n,\alpha}; n \in \omega \rangle$ of all $j_{0,\alpha}(I)$-positive subsets of ${\omega_1}^{M_\alpha}$,

- if $\alpha \leq \gamma$ in ω_1, $n \in \omega$ and ${\omega_1}^{M_\gamma} \in X_{n,\alpha}$, then $j_{\alpha,\gamma}(Y_{n,\alpha}) \in G_\gamma$, and
- for every $\alpha \in \omega_1$, $g_\alpha \leq^* f^{\alpha+1}_{{\omega_1}^{M_\alpha}}$ $(= f^{\omega_1}_{{\omega_1}^{M_\alpha}})$, where for each $\alpha \leq \omega_1$, we write
$$j_{0,\alpha}\left(\langle f_\xi; \xi \in {\omega_1}^M \rangle\right) = \left\langle f^\alpha_\xi; \xi \in {\omega_1}^{M_\alpha} \right\rangle.$$

(We note that if $\alpha \leq \beta$ in $\omega_1 + 1$ and $\xi \in {\omega_1}^{M_\alpha}$, then $f^\alpha_\xi = f^\beta_\xi$.)

This can be done by the following claim:

Claim *Assume that we have constructed $\langle M_\gamma, G_\beta, j_{\gamma,\delta}; \beta < \gamma \leq \delta \leq \alpha \rangle$ and $Z \in \left(\mathcal{P}\left({\omega_1}^{M_\alpha}\right) \setminus j_{0,\alpha}(I)\right)^{M_\alpha}$. Then there is a $\left(\mathcal{P}\left({\omega_1}^{M_\alpha}\right) \setminus j_{0,\alpha}(I)\right)^{M_\alpha}$-generic filter G_α with $Z \in G_\alpha$ such that $g_\alpha \leq^* f^{\alpha+1}_{{\omega_1}^{M_\alpha}}$.*

Proof of Claim. We have to notice that

- in a generic extension of M_α with $\left(\mathcal{P}\left({\omega_1}^{M_\alpha}\right) \setminus j_{0,\alpha}(I)\right)^{M_\alpha}$, $f^\alpha_\xi \leq^* f^{\alpha+1}_{{\omega_1}^{M_\alpha}}$ holds, hence $f^{\alpha+1}_{{\omega_1}^{M_\alpha}}$ $\leq^*$-dominates all slaloms in $\mathcal{S} \cap M_\alpha$, and
- for each $p \in \mathbb{D} \cap M_\alpha$, the set
$$\left\{\xi \in {\omega_1}^{M_\alpha}; \mathsf{body}(p) \leq f^\alpha_\xi\right\}$$
is $j_{0,\alpha}(I)$-positive.

(We note that $f^{\alpha+1}_{{\omega_1}^{M_\alpha}}$ is in $M_{\alpha+1}$ which is a subuniverse of $M_\alpha[G]$ and it is not changed by the transitive collapse and the relation $\leq^*$ is absolute.) So by Lemma 2.5, we can find a desired G_α. ⊣

By the construction (and the standard argument, e.g. [11, Lemma 4.36.] or [7, 2.8 Lemma]), $j_{0,\omega_1}(I) = J \cap M_{\omega_1}$ and $j_{0,\omega_1}\left(\langle f_\xi; \xi \in {\omega_1}^M \rangle\right)$ is a scale. What we need to check is J-goodness of the scale.

To see J-goodness, take any $p \in \mathbb{D}$. Then there is $\alpha \in \omega_1$ such that $\mathsf{body}(p) \leq^* g_\alpha$, so we can find $n \in \omega$ such that $\mathsf{body}(p) \leq^n f^{\alpha+1}_{{\omega_1}^{M_\alpha}}$. Let $g \in \mathcal{S}$ be such that $g := (\mathsf{body}(p) \restriction n)^\frown f^{\alpha+1}_{{\omega_1}^{M_\alpha}} \restriction [n, \infty)$. We note that g is in $M_{\alpha+1}$. Since
$$M \models \text{“} \langle f_\xi; \xi \in {\omega_1}^M \rangle \text{ is an } I\text{-good scale ”},$$
by elementarity of $j_{0,\alpha+1}$
$$M_{\alpha+1} \models \text{“} j_{0,\alpha+1}\left(\langle f_\xi; \xi \in {\omega_1}^M \rangle\right) \text{ is an } j_{0,\alpha+1}(I)\text{-good scale ”}.$$

Therefore the set

$$\{\xi \in \omega_1{}^{M_{\alpha+1}}; g \leq f_\xi^{\alpha+1}\}$$

belongs to $M_{\alpha+1}$ and is $j_{0,\alpha+1}(I)$-positive. Since $j_{0,\omega_1}(I) = J \cap M_{\omega_1}$ and

$$\begin{aligned} j_{\alpha+1,\omega_1}\left(\{\xi \in \omega_1{}^{M_{\alpha+1}}; g \leq f_\xi^{\alpha+1}\}\right) &= \{\xi \in \omega_1; g \leq f_\xi^{\omega_1}\} \\ &\subseteq \{\xi \in \omega_1; \mathsf{body}(p) \leq f_\xi^{\omega_1}\}, \end{aligned}$$

the set $\{\xi \in \omega_1; \mathsf{body}(p) \leq f_\xi^{\omega_1}\}$ is J-positive. $\square$

We can show the strategic iteration lemma for the existence of a good scale using arguments of the previous proof and [10, Lemma 2.8.]. So we can conclude Shelah–Zapletal's theorem that $\mathfrak{d} = \aleph_1$ is Π_2-compact.

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